SLIM: A Slender Technique for Unbounded-Field Problems

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***Abstract*— Electrostatic- and electromagnetic-field problems in unbounded regions are often solved using finite differences (FD's) or finite elements (FE) combined with approximate boundary conditions. Inversion of the sparse FD or FE matrix is then required. First-order or higher order absorbing boundary conditions may be used, or one can use more accurate boundary conditions obtained by the measured equation of invariance (MEI) or by iteration. The more accurate boundary conditions are helpful because they permit reduction of the size of the mesh and, thus, the number of unknowns. In this paper, we show that the process can be carried to a maximally simple limit in which the mesh is reduced to a single layer and the matrix-inversion step disappears entirely. This results in the single-layer iterative method (SLIM), an unusually simple technique for unbounded-field problems. Computational experiments demonstrate the effectiveness of SLIM in electrostatics, and also in electrodynamic examples, such as scattering of TM plane waves from a perfectly conducting cylinder. The technique is most likely to be useful in large or complex problems where simplification is helpful, or in repetitive calculations such as scattering of radiation from many angles of incidence.**

***Index Terms*— Adaptive estimation, boundary-value problems, finite-element methods, finite-difference methods, scattering.**

I. Introduction

When mesh-based methods such as the finite difference (FD) method are used in unbounded regions, it is often useful to terminate the mesh through the use of an approximate boundary condition. One well-known technique for doing this makes use of absorbing boundary conditions [1], [2]. In its simpler forms, this method simulates the external fields that are truncated from the problem by assuming that they have the same form as the fields of a simpler radiating structure, such as a radiating dipole. In order to obtain accuracy, the truncation must typically take place at some distance from the radiating structure, which results in a large mesh, with the associated computational requirements. Mei *et al.* [3], [4] introduced another technique for mesh termination, known as the measured equation of invariance (MEI) technique. In MEI, approximate conditions on the fields at the boundary of the mesh are obtained by using Green's functions representing the fields of charges or currents on the conductors. These are combined with conjectural charge or current distributions ("metrons") on the conductors to compute conjectural fields at the mesh boundary, from which boundary conditions are derived. Interesting work has been done on the mechanisms and accuracy of this method [5]–[7]. In many cases, it is found to work surprisingly well. However, in general cases, its accuracy is limited because the boundary conditions are based on current distributions which are not the correct ones, but only guesses. Unfortunately, it is difficult to estimate how much error will arise in a particular case.

A further step was taken by Li and Cendes [8] and Jin and Lu [9], who applied what they referred to as "adaptive" boundary conditions. Their approach is similar to that of Mei *et al.* in that it reduces computational effort by truncating the FD or finite element (FE) mesh close to the conductors or scatterers. However, in the adaptive method, an initial incorrect guess of the charges or currents is corrected through iteration. That is, one guesses the currents and then calculates the resulting fields at the boundary of the mesh through the use of Green's functions. From these fields, one calculates boundary conditions at the boundary of the mesh. Then, it is possible to recalculate the fields and, hence, the currents by using the approximate boundary conditions. The process can be repeated and is found to converge. If convergence does occur, the resulting fields and currents satisfy the boundary conditions and Maxwell's equations and, thus, should be correct within computational error.

The adaptive technique is more subtle than it first appears. One might think of a similar, but simpler technique in which one would (using, for simplicity, an electrostatic example) try to find the field of a charged conductor by guessing a charge distribution, finding the electrostatic potential at the boundary, applying this potential at the boundary to create a Dirichlet boundary condition, and then calculating the fields by FD's, from which one would then find a corrected charge distribution. However, in many cases, this method is found to diverge. The corresponding approach of calculating the normal field at the boundary and iteratively solving the Neumann problem is even less likely to converge. A more successful approach is that of finding the ratio $a(h)$, defined as the potential at each point A on the boundary divided by the potential at point B, which is the nearest neighbor interior point, separated from A by the lattice spacing h. One then applies the boundary condition $V_A = aV_B$ and proceeds with the FD solution. If we define $a(h) = 1 + Rh$, then to first order in h, the boundary condition being applied is

$$\frac{\partial V}{\partial n} - RV = 0 \tag{1}$$

where R is a function of position. Interestingly, this boundary condition (which we shall refer to as a "ratio boundary condition") is neither the Dirichlet nor the Neumann boundary condition, but something which combines the two. For a physical interpretation of this condition, we note that on a one-dimensional transmission line, the voltage V and current I obey $\partial V/\partial z = -j\omega LI$. Thus, in the one-dimensional case, specifying R amounts to specifying the admittance with which the transmission line is terminated. In higher dimensions, the ratio condition implies specification of the normal impedance at the boundary. Use of the ratio condition in the iterative method described above is usually found to lead to convergence, although to our knowledge a general proof of this is lacking. Mathematical interpretation of this boundary condition is made difficult by the product term RV. It generates new spatial frequencies not present in either R or V.

A further step was taken by Rius *et al.* [10], who converted the original MEI technique into a sparse matrix boundary-element method. In this method, the FD mesh collapses onto the scattering surface, and all calculations take place on that surface. Solution of a sparse FD matrix is still required.

In this paper, we point out that the methods just described can be combined to yield an adaptive technique in which the FD calculation disappears entirely, and no matrix inversion is required at all. Using an electrostatic example, the approach is as follows. Fig. 1 shows an air-spaced microstrip structure with ground-plane and a conductor held at potential V. Surrounding the conductor (but not the ground plane) is a single layer of mesh points at which the potentials v_i^+ and v_i^- are unknown. We guess an initial charge distribution $q_j^{(1)}$ and use it with the appropriate Green's functions $G(r_i, r_j)$ (which include the effect of the ground plane) to find a set of trial potentials. For

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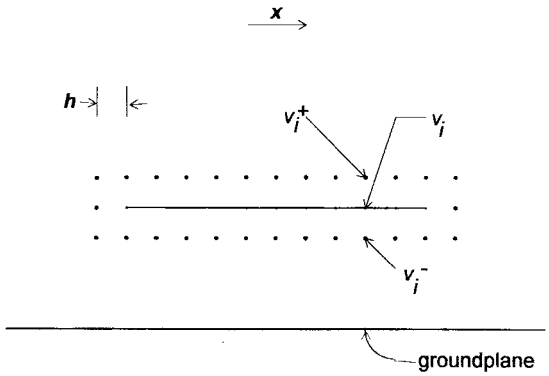

Fig. 1. Air-spaced microstrip structure. The FD mesh is a single layer of points enclosing the strip (but not the ground plane).

each position on the conductor, we calculate trial potentials at three positions: v^+ and v^- at positions just above and below the conductor, and v at the point on the conductor that lies between them. These trial potentials are then used to obtain the ratios $a^+ \equiv v^+/v$ and $a^- \equiv v^-/v$ at each position on the conductor. The ratios are then used to find a first estimate of the potentials using $v_i^{+(1)} = a_i^+ V$ and $v_i^{-(1)} = a_i^- V$. The second estimate of the charges is then from Gauss' Law, $q_j^{(2)} = (2V - v_j^{+(1)} - v_j^{-(1)})/\epsilon h$, where ϵ is the permittivity and h is the lattice spacing. (Appropriate modifications are made at the edges of the conductor.) The sequence of steps is then repeated until $q^{(n+1)}$ agrees with $q^{(n)}$ within required accuracy. This method, which we call the single-layer iterative method (SLIM) is found, in numerical experiments, to work for some important classes of problems. For example, it seems to work well in electrostatics with air-spaced microstrip patches of any shape. Interestingly, it is also found to work in electrodynamics, for scattering of TM plane waves from a perfectly conducting cylinder. On the other hand, some cases have been found in which erratic or even divergent solutions are obtained, and this requires investigation.

It is by no means obvious that solutions of the boundary-value problem with ratio boundary condition are unique. We have succeeded in proving this only for the electrostatic case, under the assumption that all conductors, except the ground plane, are at the same potential. As regards convergence, we have demonstrated something like local convergence in electrostatics, by showing that if the error in a charge distribution (difference between it and the correct distribution) is small, the error of the charge distribution that results from the next iteration approaches zero in the limit as $h \to 0$.

II. Computational Examples

As a first example, we apply SLIM to an air-spaced microstrip structure without time variation. The result after ten iterations is shown in Fig. 2. The conductor is held at a potential of 10 V, and the ground plane at zero. Here, the circles are from SLIM, while the solid line is found from the method of moments (MOM). Agreement is close, except at the endpoints. The endpoints are singular, and neither method can be expected to be meaningful there. However, we also note that in SLIM it is necessary to calculate the charges from the fields, a step that is unnecessary in MOM. This step can give difficulty at singular points and might lead to extra errors.

When we turn to electrodynamics, there is little theory available to assure convergence or uniqueness. In computational experiment, however, the method is generally found to work. A SLIM calculation of scattering of a TM plane wave by a perfectly conducting cylinder with diameter equal to wavelength is shown in Fig. 3. In this calculation, the unknowns are the z-directed currents at the lattice

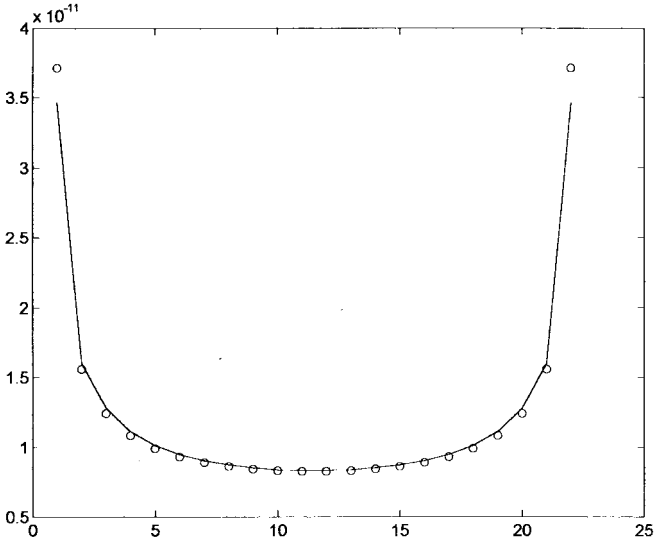

Fig. 2. Calculated charge density on air-spaced microstrip held at 10 V, in farad per meter. Circles are from SLIM (ten iterations); solid curve from MOM. The width of the strip is 21 h and the distance between strip and groundplane is 20 h.

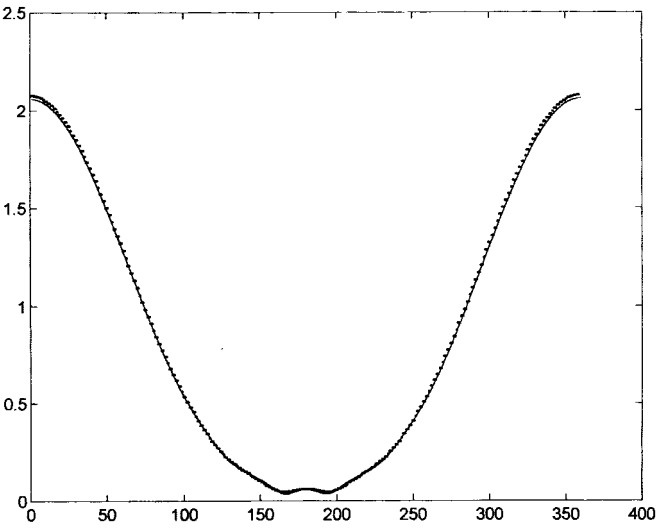

Fig. 3. Scattering of plane wave by perfectly conducting cylinder (electric field parallel to cylinder axis.) Graph shows current density as a function of angle in degrees. Diameter of cylinder is equal to wavelength; h is equal to circumference divided by 360. Solid curve from SLIM (seven iterations); dotted curve from series solution.

points on the circumference of the cylinder. Using the well-known expression for the fields of a sinusoidal current in an infinitely long wire [11], the electric fields E and E^+ arising from a guessed initial current distribution are calculated at lattice points, respectively, on the surface and in a single layer enclosing the surface. The scattered field at the surface is known to be the negative of the incident field at the surface. The new estimate of the scattered field at each lattice point above the surface is then taken to be the product of the ratio (E^+/E) at that position times the known scattered field at the point directly below it on the surface. The difference between the new estimate of the scattered electric field above the surface and the known field on the surface is then used to find H_φ, from which the new value of the current at that position is obtained. The solid curve in Fig. 3 is the z-directed current density as a function of angular position, after seven iterations. The lattice spacing is 1/360 times the

circumference. The dots are from the well-known solution in a series of cylindrical harmonics. Agreement is seen to be very good, even including the small "bump" at 180°. Convergence depends on use of a sufficiently small lattice spacing compared with wavelength. A rough experimental maximum for h is $\lambda/8\pi$.

III. Discussion of the Method

SLIM is a technique for solution of unbounded electrostatic (and conjecturally, electrodynamic) field problems that requires no inversion of matrices. This makes the method simple to program and use. Its simplicity should be especially helpful in otherwise complex problems involving oddly shaped boundaries and three-dimensional fields.

The method's characteristic self-consistency also tends to create confidence in the accuracy of the results (as compared, for instance, with ordinary absorbing boundary conditions, which give results whose accuracy is unknown). If one finds a set of charges and fields which are self-consistent (in the sense that the fields imply the charges and vice versa) and which satisfies the boundary conditions, it seems quite likely that this solution will be correct. On the basis of intuition and experience, we conjecture that in all cases in which SLIM converges, the results are correct to within accuracy of computation.

An important question is whether SLIM actually offers any saving of computational resources as compared with other methods that require matrix inversion. MOM is undesirable for large problems because it results in full matrices, inversion of which requires on the order of N^3 operations (where N is the number of unknown charges or currents). FD or FE methods with various absorbing boundary conditions (or the MEI boundary conditions) result in sparse matrices requiring on the order of only N^2 computational operations. In comparison, SLIM eliminates the matrix-inversion steps and reduces the storage of unknowns to minimal size. However, one must still evaluate on the order of N^2 Green's functions: once for each field point for each source point. Depending of the complexity of the Green's function, this step may take longer than the matrix-inversion step, which makes the elimination of that step less important. On the other hand, the evaluation of the Green's functions only has to be done once for each structure. Once the Green's functions have been found and stored, further computational steps can take place quickly. Thus, the second and subsequent iterations should require only on the order of N operations. Moreover, in scattering problems, the Green's functions are characteristic of the scatterer, not of the illumination. Thus, if one is finding radar cross sections with many angles of illumination, all the angles after the first one should require computational steps on the order of only the first power of N.

As in any technique in which simplicity and lightness are pushed to the limit, a question of reliability must arise. It is quite possible that SLIM is less robust than the adaptive absorbing boundary condition techniques (with matrix inversion) described by Jin and Liu [9]. This question requires further study.

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Using Selective Asymptotics to Accelerate Dispersion Analysis of Microstrip Lines

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***Abstract*— A selective asymptotic technique (SAT) to accelerate the elements of the impedance matrix in the conventional spectral-domain approach (SDA) is presented. Instead of using the full asymptotic expression of the Green's functions, only those parts which cannot be evaluated in closed form are approximated by their asymptotic expressions. The resulting expressions are more accurate and systematic, as no additional parameter is introduced. The technique is applied to determine the effective dielectric constant of an open microstrip line to demonstrate its efficiency.**

***Index Terms*— Dispersive media, microstrip lines, numerical integration, spectral-domain approach.**

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